Gödel's incompleteness theorem in mathematics teacher formation courses: previous possibilities

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Abstract: *In this paper, we present our perspective about the importance of teaching Gödel's incompleteness theorem in formation courses to mathematics teacher. The phenomenon of incompleteness evidenced by the incompleteness theorem is manifested by the existence of true propositions about natural numbers that cannot be proved by any metamathematical argument that can be represented according to arithmetic formalism. Knowing Gödel's incompleteness theorem and its conclusions is essential in terms of the mathematical culture of a teacher in preparation to teach mathematics in schools, so that he/she can avoid nurturing the notion of complete mathematics and the notion of axiomatic method totalitarianism. This proposal seeks to evidence that Gödel's incompleteness theorem can be explored in formation courses of mathematics teachers[1] in exercises in Philosophy of Mathematical Education.*

INTRODUCTION

In our trajectory of undergraduate teachers and researchers whose goal is to form mathematicians and mathematics teachers, we have faced several debates about the scope of professional practice in Mathematical Education. We understand that in the curriculum itself of these courses it is possible and desirable to insert indications for discussions about mathematical contents themselves, as well as the related philosophical questions.

One challenge we have faced in working with Philosophy of Mathematics and Philosophy of Mathematical Education is the predominance of structuralist and formalist conceptions of Mathematics. The view of Mathematics present in these conceptions contributes strongly to the

[1] We do not call these courses by the name "mathematics teacher training courses" because we do not agree with the term "training" which is derived from the verb *train*. Training is a term whose meaning refers to physical exercises, aiming the action of regularly performing a physical activity with a view to competitions, for example. One can also successfully train animals, e.g. Pavlov's experiments. This does not involve fundamental concepts to the profession of being a teacher like *thinking* or *reflecting*.

future mathematics teachers and mathematicians focus primarily on knowing how to work with mathematics and operate with its tools, moving far away from the aspects of history and modes of production of this science, and also is not questioned the social practices that affect this production.

As we understand, formation courses to math teachers need to work in an articulate and interdisciplinary way with mathematical character themes, bringing specific contents of items seen as important for the student to build knowledge about mathematics in its various disciplines and ways of application. At the same time, it is necessary to work with logical and language structures that are at the core of this science production, and to focus on important themes for understanding the complexity of education seen its broadest scope, and in a more specific way, in that of the school education, addressing the surrounding administrative and socio-cultural aspects.

Based on this view, when working in courses that focus on the formation of mathematics teachers in Brazilian universities[2], we are concerned with working in a daily range of issues we consider significant when viewed in the intended horizon. We take as material to be understood and questioned the mathematical information taught in different subjects of the curriculum, questioning them about what they say, what are the essential mathematics ideas that underlie them, what is the scope of their possible applications, what are the structures present in their production, what is its historical horizon. The questioning attitude we take with students and mathematics focuses on the way they begin to look at mathematics itself and also at themselves as learners and future teachers of this science. Lins (2005) argues that a teacher:

> (...) needs to know more, not less Mathematics, but always clarifying that the more does not refer to more content, but to an understanding, greater clarity, and this necessarily includes the understanding that even within the mathematician mathematics, we produce different meanings for what seems to be the same thing. (LINS, 2005, p. 122)

Teacher education perspective presented in Bicudo (2010) underlies the possibility of working with Gödel theorem in mathematics teacher formation courses that we present. That perspective is exposed in the item bellow.

FORMATION OF MATHEMATICS TEACHERS

We understand that future mathematics teachers need to comprehend mathematics beyond knowing “how to do” and familiarize themselves with the issues of Philosophy of Mathematics in

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order to understand meanings of accuracy and certainty present in this science. We understand that working with Gödel's incompleteness theorem in formation courses opens possibilities for raising epistemological questions concerning the production of proof of the theorem and also of Logic.

We consider that mathematics teacher formation courses, by providing opportunities for students to understand the way mathematical theories are constructed, require them to provide technical, analytical and reflective thinking on/about some of these activities. Thus, important issues of Mathematics and its structure are being discussed and those can be taken responsibly in the practice of Mathematics Education of teachers, as they get to know the updated view of Mathematics and not only its disconnected parts.

According to Bicudo (2010), the proposal to form teachers is shown as a project, understood in the heideggerian sense, as a project, in which "pro" says to put it ahead and "to throw" says to launch the happening to the possibilities of being what was outlined as a formation proposal. In the theme focused here: it means to throw the person in the world of possibilities announced in the mathematics teacher formation project. As people make choices and update actions, styles of their way of being a teacher are gradually outlined. Thus, formation occurs in a continuous flow of actualizing the possibilities of the project, whose sense of being is in correspondence with the context of the surrounding world that welcomes it and from where it arises.

There is a historical and social ground that brings a culture imaginary regarding to be a mathematics teacher and that gives direction to the movement. This imagery encompasses the aspirations, usages and customs of peoples, their codes of honor, appreciated values, as well as the force that moves people toward the perception of duties and makes them proud of their achievements. It is not a matter of imprisoning this imaginary in a goal to be reached, as if it were objectively given. However, it is a direction that refers to the actions ahead. Along with these forces are the sociocultural demands of the socially organized community that are imposed on the movement of outlining the project. Still, in the worldhood of the world there are facticities to be faced to make the project happen. What we are saying here is that the proposal understood as a project does not narrow in linear sequences of predicted occurrences, seeking also predicted results. Instead, the project is opening possibilities for teacher formation to happen. It is a movement in which the actualizing of becoming:

> [...] is effected with what moves, and what moves also has its strength, which means that the form cannot conform to the action, but the action itself, acting with the matter establishes the form upon it. There is, therefore, a play between ideal, understood as a form that

> establishes direction, action, driven by the prevailing force that vigorously impels the person to an act, and which emerges from a sense of duty and pride, for having succeeded in becoming what one becomes, and matter constituted by the reality of people's lives, encompassing their historicity, their myths, their ways of warning, imposing precepts, communicating knowledge and professional skills. (BICUDO, 2003, p. 31).

The conception of teacher formation we assume is understood as a continuous and uninterrupted movement of becoming in what the *form* ideas pertinent of being a mathematics teacher, the *action* triggered by the willingness of a subject to do something in this direction, the *materiality* available to, along with the form, materialize the act comes together in a movement of *becoming*. In other words, from an ever-moving mathematics teacher formation working mathematics and its modes of production, to the people involved in the acts of teaching and learning, to the facticities of the life-world of school and society, science, technique and technology at their disposal. We understand, therefore, that working with Gödel's theorem in formation courses opens the possibilities for students to understand the very construction of mathematics, from the point of view of the logical enchainment of a theory and the demonstrations within the framework of theories, the ideas that drive this enchainment, the aspects of reality (ontology) of mathematical objects.

GÖDEL'S INCOMPLETENESS THEOREM IN TEACHING

Since 2013 we have been committed to know deeply Gödel's Incompleteness Theorem(s) (GIT) and to seek the coherence and possibilities to implement them in mathematics teacher formation courses, Batistela & Bicudo (2018). We understand both historical and epistemological meanings of this/these theorem(s) as well as the pedagogical possibilities of, through the approach of the theorem demonstration, highlighting the deep connections that link Logic with Mathematics and articulating strong arguments that support this congruence. In this perspective, two expectations are announced. The first is to explore and epistemologically problematize the implications of Gödel's Incompleteness Theorem for understanding aspects of Mathematics linked to anthropological possibilities of man *being-in-the-world* with others in relation to mathematics teacher formation. The second, together with this, seeking to work with mathematical knowledge logical foundations, aiming to create possibilities for students to realize the process of constitution and production of knowledge of objects involved in teaching and learning Mathematics.

It is important to emphasize that there are two[3] Gödel's theorem(s). Nagel & Newman (1973) understood as having clearly differentiated parts in which each one, with its individual importance. Those parts refer directly to the intimate relationship between Logic and Mathematics. This relationship manifests itself in a notorious way in the so-called "formal axiomatic method" which, duly appropriately from the nineteenth century, permeates various current mathematical theories, emphasizing the underlying logic and the language in which they are expressed. The origin of the current axiomatic method, formal for the purpose of its logical analysis and material for its use in mathematical practice, is connected to Euclid systematization of geometry in antiquity.

The research presented by Batistela (2017) enabled us to draw up an immersion plan for Gödel test. With this we organized a script of activities related to the process of mapping arithmetic in metamathematics. This led us to understand questions about metamathematics as the locus in which GIT was proven and to reflect on the impact of this result, especially on the conception of knowledge in mathematical science:

> Metamathematics is a field of logic that tries to demonstrate results about mathematics. It differs from mathematics whose objects are numbers, geometric figures, functions, etc. and not mathematical sentences. It is therefore a meta-language of mathematics. Thinking of mathematics as a language about something, metamathematics is a language of a language, which some philosophers (such as Russell and Wittgenstein) define as a higher-level language. (LANNES, 2014, p. 5, footnote number 5).

GIT is a Mathematical logic theorem that introduces knowledge of mathematical logic theories into unresolved problems of mathematics at that time and that were in focus from the turn of nineteenth to twentieth century. Incompleteness theorem provides proof that the consistency of arithmetic cannot be demonstrated in arithmetic itself. This problem of the consistency of arithmetic had been attacked by important and competent mathematicians of that time, but during that historical moment there was only relative evidence about that consistency. An absolute proof

[3] The first can be stated thus: "If arithmetic is consistent then G is not demonstrable." Where G is a true formula of arithmetic. In other words, he says that in formalized arithmetic there is a formula that is true but undecidable. The second: "If arithmetic is consistent then A is not demonstrable." Where A is an arithmetic formula that represents the metamathematic proposition: "Arithmetic is consistent." In other words, he argues that if arithmetic is consistent, its consistency cannot be proved by any metamathematical argument that can be represented in arithmetic formalism.

to arithmetic consistency was at the core of mathematicians' endeavors, and one believed that it was certain, but to obtain it was only a matter of time.

We believe the occasion for discussion and work with Gödel's incompleteness theorem demonstration is an opportune moment to think together with students about the philosophical posture implicit in Formalism and to reflect on GIT message for this current of mathematical production and, consequently, for Mathematical Education production. It is well known that GIT has no direct relation with the contents taught in Primary Education (understood as the work with children until fourteen years old) but one of the consequences of its conclusions elucidates a limit in Mathematics formalization. This understanding may contribute to conceiving Mathematics as not strictly exact science. GIT confirms the existence of a fundamental characteristic related to the way one practices mathematics. We understand this is why it relates directly to the way we comprehend mathematics and therefore to the way we teach it. We believe that the experience with GIT makes possible for mathematics students in teacher formation courses to become aware of the incompleteness of mathematics and the importance of this result as it has produced a change in mathematics own conception.

By assuming this proposal through formal ways - working with an illustration of Gödel's incompleteness theorem demonstration - we admit we can raise a reflective thinking about mathematics conception in which formation course students are immersed. Thus, we can link possible conceptions that are present in current culture of mathematical science since Gödel's theorem exposes the reach of mathematics, pointing out the limits of formalism and differentiating mathematical truths from demonstrability.

The predominance of formalism and structuralism in mathematics teacher formation courses is well known. Among the three philosophical currents that most influenced mathematics, its production and consequently its teaching, Hilbert's Formalism is the one that stands out, although this was the current that was affected the most by GIT. According to Wittman, "Although Hilbert's dream burst already in 1930 when Gödel provided his incompleteness theorem, the formalistic setting of Hilbert's programme has survived and turned into an implicit theory of teaching and learning." Wittman (2001, p. 6). One of the most common forms of structuralism in undergraduate courses always appears when the origin of the historical sense of mathematical knowledge is concealed, that is, when the context of discovery, probing or research is supplanted by the logic of best exposure with formal justification.

In the case of the philosophical schools that sought to ground mathematics, besides Formalism there was Logicism and Intuitionism. These philosophical currents admit, respectively, that it would be possible to translate all mathematical expressions into logical expressions and that all the most elaborate mathematical objects could have their existence demonstrated by constructive proofs, that is, by not admitting demonstrations by reduction to the absurd.

At the public presentation of GIT, Logicism and Intuitionism were projects with limitations and Formalism was on the march with the project of axiomatizing all Mathematics through Logic extirpating the semantics of mathematical discourse and making Mathematics as pure manipulation of symbols, that is, a formal symbolic system. There was agreement among mathematicians that the idea from Intuitionism, namely, that finitism from natural numbers was the basis of Mathematics. Hilbert knew and agreed to prove that all mathematical demonstrations could be performed by deriving directly, by finite steps, from Peano axioms. If this were the case, the symbolic systems using Peano axioms would be consistent, and it was only necessary to prove that they were a consistent set of axioms, meaning that the objects defined by them exist.

It is precisely at this point that Gödel's theorem has an impact. The incompleteness of Peano Arithmetic and the impossibility of demonstrating within Peano Arithmetic that a true formula in the language of Peano Arithmetic is true and that this theory is not consistent.

The work with GIT that we conduct in mathematics teacher formation courses goes beyond the study of the demonstration performed by Gödel that performs the mapping of metamathematics in Arithmetic. Building this mapping, Gödel uses the same bases proposed by Hilbert in the Formalism Program, i.e., the axiomatic system for numbers is based on deductive inference rules, which one can define the demonstrativeness itself of an expression within the formal theory of numbers. In the first part of the demonstration, the one that will establish the result of the first incompleteness theorem, a sentence is constructed in the language of Peano Arithmetic which represents that a formula φ is demonstrable by the T theory.

The formula is "If formula φ is demonstrable in theory T, then formula φ is true," that is, it can be formally represented in the language of Peano Arithmetic, expressing the correction itself. Hilbert Program hoped to prove that if Peano Arithmetic were a correct system it would be consistent, that is, it believed in the equivalence between correctness and consistency of a system. A system is said to be correct if all the axioms and truths of the system are demonstrable and true - if so, the deductive calculation is correct in first-order logic.

As Gödel built his demonstration on the same basis as proposed by the Hilbert Program, so this result establishes that this Program was predestined for failure.

In the proposal of teaching GIT in teacher formation courses, one takes into account subjects from previous course disciplines that support the understanding of the details of the test and the development of the argumentation that directs the established conclusions. The intelligibility of the parts of the theorem - that pass through the numbering of symbols, formulas, sequences of formulas, by the system that produces Gödel numbers up to the construction of the undecidable formula as well as the second part in which the logical argumentation - states that the undecidable formula is the one that states that if arithmetic were consistent then it cannot prove its own consistency. In addition to the parts, the message that it constitutes and conveys is also treated by the discussion of the ideas present in the demonstration by the repercussion of this result in the mathematical community.

In the work with GIT in mathematics teacher formation courses, students attest to understand the construction and the idea of the undecidable as a truth that doesn´t need to be proven and, in the context of demonstration, they see it as an exception to the general rule. The limitations established by the certain existence of undecidables demonstrate that they exist in basic natural arithmetic and in all related formal systems. The statement about these limitations is based on the understanding that mathematics contains in itself indemonstrable truths. This idea contributes to the understanding of the difference between truth and demonstrability, something important to be worked with students of this science.

The students who would experience the GIT teaching proposal express their understanding of the limitation of axiomatic method, as there are truths cannot be demonstrated. The first part of the illustration of GIT demonstration works the construction of the undecidable formula is understood by the students to the point that they can talk about it. The second part of the demonstration needs more reinforcement to open for students' horizons of understanding about the undecidable. This is because it is necessary to understand the need for proof of an object for it to exist mathematically, according to our comprehension articulated to the above on the understanding of Gödel's numbers and the undecidable. Students express the undecidable as being something that "doesn´t need to be proved" articulating with the understanding that Gödel's numbers differentiate false propositions from true propositions. As the undecidable is a true statement, deduced by a metamathematics argument, what is evident is that the theory has the possibility of deducing truths and the theory itself cannot possibly prove them all. This understanding is different from the idea that being true does not need to be proved.

Mathematical educators express their understanding that Gödel's incompleteness theorem is inadequate to be part of the subjects discussed in undergraduate courses. In our teaching activities with GIT, undergraduate students show interest and remain aware of the mathematical and philosophical themes worked through the classes. We noted that students, who were previously unaware of the existence of Gödel's incompleteness theorem and undecidable(s) in Peano Arithmetic, which implies a limitation of the axiomatic method, value this knowledge and reflect on it, understanding mathematics vividness as a whole. From their expositions, one understands that they move away from the naïve ideas that take mathematics as a sovereign and complete science.

From the statements observed in students' works, we understand we cannot avoid the discussion about the presence of the Philosophy of Mathematics and the Philosophy of Mathematical Education in undergraduate formation courses. Most of the time, the criterion adopted for curriculum changes is the usefulness of the knowledge to teachers when working in the classroom. Following this view, the course reflects, in its curriculum, the understanding that the subjects aim to add contents that will be directly used by teachers when working in Primary Education. The criterion of direct utility of contents is, therefore, what ends up being elected. Many undergraduate students from teacher formation courses finish their courses with the idea that contextualizing mathematical subjects culturally and/or socially is sufficient for primary school students to be interested and thus perform well in mathematics. Thus, they reinforce the vision that teaching aims the immediate use of contents.

This atmosphere of appreciation of productions that have direct and immediate utility has been valued in Brazil for a long time and is reflected in the curricula of Primary School, removing the obligation of disciplines like Philosophy and Sociology, and reducing the course load of other components, for example. Based on this same reasoning, federal, state and municipal governments, most of the time, decide on investments in education.

It is not our focus here to discuss the hierarchy of scientific productions usefulness or disciplines in undergraduate courses that best serve basic education. Our defense of teaching Gödel's theorem, as we have already announced in this work, is mainly about understanding the axiomatic method by which all mathematics, understood as a science of Western civilization, and even that one taught in Primary School, is built.

THE STUDENTS AND GÖDEL'S THEOREM IN FORMATION COURSE OF MATHEMATICS TEACHERS

In this topic, we aim to describe the challenges of the inclusion of Gödel's theorem in a formation course of mathematics teachers[4], as well as how the futures teachers reacted to the teaching of this theorem.

The inclusion of Gödel's incompleteness theorem in a formation course for mathematics teachers took place as a topic in the curriculum. In this way, we avoid expanding the number of subjects in the curriculum, as it already includes a discipline that provides the contents of History and Philosophy of Mathematics.

On the first day of class, this topic was presented and discussed with the students. It was exposed the importance of knowledge of Philosophy – especially Philosophy of Mathematics – for a mathematics teacher.

In 2018, the curriculum was adapted to meet the requirements of the Conselho Nacional de Educação - CNE (National Education Council). Curriculum decisions are made within the scope of the courses' collegiate bodies. The vision that guided the courses that were included in the curriculum aimed to deal with the mathematics that the mathematics teacher must know to teach it in the Basic Education school. The value that supported the criteria adopted for the changes was "usefulness of this knowledge for the teacher" when working in the classroom. Thus, the course ends up reflecting, in its curriculum, the comprehension that the subjects propose to add contents that will be directly used by the teacher when working in Basic Education. This stance responds to what we explained about the vision of a formation course focused in the immediate use of the contents.

The discussion on the insertion of the GIT – presenting students with the consistency of the treatment of this theme, in view of the subject's syllabus and its development – aimed to clarify that it is a subject not included in the curriculum. This fact raised questions about the reasons why the subject would be worked. Then, a discussion happened among the students. Some opposed to this study, as it would not be useful; others, however, expressed themselves in favor, as they understand that mathematics develops within the scope of responding to its internal problems and does not have the main goal of being applied in some area of knowledge or in everyday life.

[4] This experience with the students was lived by the first author of this article, however, the critical analysis was carried out by both authors. This activity was developed at Feira de Santana State University, in Bahia, Brazil, specifically in the area of Mathematical Education. Most students work as teachers in the public school system.

Another interesting point that was raised in the discussions (and in the texts they produced during the course) concerns the fact that most students refer to and agree with a metaphor of their knowledge, elaborated by Batistela (2014). The metaphor says, in short, that an airplane pilot needs to know more about the functioning of the airplane than the passengers, much more than what he/she can communicate through the airplane's speakers about the height of the flight and the expected time of arrival. In the metaphor, there are relations of similarity between the teacher and the pilot, mathematics and the plane, the teaching of mathematics and the speeches of the pilot during the flight, the mathematical knowledge that the mathematics teacher needs to master, mainly regarding the nature of mathematics and the range of mathematical production methods, and the knowledge that the pilot needs to master about the operation of the airplane, the rate of fuel burn, the distance to be covered, the amount of fuel he/she needs to store. A pilot does not need to tell passengers that the plane is not able to go around the world and return to the same airport because the fuel tank does not have the capacity and that the production of a plane with a fuel tank with enough capacity to this would require efforts and materials that are still unavailable today.

Our intention, by exposing the coherence of the study of Gödel's theorem in the formation courses of mathematics teacher is to present the political struggles that we assume, including among mathematical educators, when proposing the teaching of a result that says about the construction method of this science and not about contents that will be directly taught in schools. In addition, it is a subject that needs technical knowledge of Mathematics and Logic to be understood. At the same time, we understand that the insertion of this theme would not harm the treatment of other matters considered important by the Collegiate and by the students.

In the classes that we conducted and in the discussions we held, the various subjects that the students had already studied were articulated, concerning the ideas that precede the emergence of the theorem, essential for the comprehension of its importance and the impact that it had. During the classes, activities were carried out on how the mathematics they study at the University is structured.

In summary, the incompleteness theorem was presented as a result of Logic that uses arithmetic and performs a mapping in metamathematics, with repercussions in mathematics. This vision guided the specific objectives of each part of the course, which addresses: 1) the historical and cultural environment in mathematical science in 1931, the year that GIT was disseminated in the mathematical community; 2) the realization of a proof of the theorem; 3) the way in which mathematics accepted this result; and 4) the way in which mathematicians currently work in terms of structuring their theories.

In the theme of the crisis of the foundations of mathematics, we present the projects and works of David Hilbert, Henri Poincaré and Luitzen Brouwer in the search for the resolution of this crisis. We made considerations about the emergence of mathematical philosophers and logicians to solve the problems with which mathematicians were working. We explained that mathematical logic was not socially determined, and that Gödel's incompleteness theorem was a landmark that caused a split between mathematics and mathematical logic, as it shed light on the specificity of each one. In this regard, Lannes (2014) states:

> Other demonstrations of this split are: the classification of areas of knowledge by government agencies (which include mathematical logic as a sub-area of philosophy), acceptance of works in congresses and specialized magazines, curricular matrices of education college (logic is commonly included in philosophy courses and excluded in mathematics courses), differentiated teaching staff at universities, separate organization of research groups and graduate programs, etc. (LANNES, 2014, p. 8)[5]

According to Lannes (2014), Gödel´s theorem and Kurt Cohen´s continuum hypothesis are the first suggested results that refer to the framework of mathematical logic and, therefore, seal the beginning of the history of this area making the mentioned split plausible.

We understand that the students listened attentively the subjects and that, for them, two ideas became clear: there should always be knowledge being sought and being built in mathematics and the idea that there are flaws or weaknesses in arithmetic. They showed that they understood that GIT brings the message that mathematics continues to exist and is under construction and that they understand that the undecidable is a weakness, in the sense that it is not possible to prove a truth. This is a key point and it affirms statements that do not match the nature of mathematics.

The construction of mathematical objects through demonstrations is clear to them, but in some aspects it is confusing, as some students claim that the undecidable proposition can still be shown to be false. On the other hand, we note that the proof of the theorem was not clear to them, especially in the second part in which one builds and develops the logical argument that leads to

[5] In the original: "Outras demonstrações desta cisão são: a classificação de áreas de conhecimento por órgãos governamentais (que incluem a lógica matemática como sub-área da filosofia), aceitação de trabalhos em congressos e revistas especializadas, matrizes curriculares de cursos superiores (a lógica é comumente incluída em cursos de filosofia e excluída em cursos de matemática), lotação diferenciada de docentes em universidades, organização separada de grupos de pesquisa e programas de pós-graduação, etc."

the undecidable. The links between mathematical logic and mathematics need to be further worked on in these courses and we believe that, then, aspects of metamathematics could be more clearly understood by students.

We highlight two points that became evident to us as having been little understood by the students: the part of the demonstration that builds the undecidable proposition and the idea that there is a truth that cannot be proved to be true - or even false. We interpret that these are aspects that demand greater maturity of mathematical thinking because they involve the deeper connections of Gödel's argument in which the undecidable is created as a truth that is not derived by the rules of the logic of Peano's axiom system, but it is true in technical sense of mathematics.

Although the students showed that they were unable to understand the message of Gödel's theorem and the logical relations of the demonstration in its scope, we can say that the treatment of this theme prompted them to think about mathematics. When we asked for their opinion on whether or not this subject should be studied in teacher formation courses and if they would like to continue studying it, they stated that working with the incompleteness theorem in undergraduate courses is a unique opportunity to learn about mathematics and sheds light on philosophical questions related to mathematical logic, as well as about logic and mathematics. Their answers show that they would like to enter into philosophy studies focused on mathematics and mathematical logic themes.

Significantly, the students' responses reveal that they understand that the study and reflections carried out when approaching GIT can indeed contribute to the teaching of mathematics in schools. His arguments in favor of this idea go in the direction of the dominant conception of mathematics. They argue that, to the extent that the mathematics teacher understands the potentials and limitations of mathematical thinking, he/she can demolish illusions about the accuracy and sovereignty of this science in relation to other areas of knowledge.

As a result of the statement that mathematics contains indemonstrable truths in itself, which proved to be difficult to understand, we chose to discuss the difference between truth and demonstrability. From their exhibitions, we comprehend that, by understanding them, they move away from the naïve ideas with which they had been working when they accepted that this science was exact, sovereign and complete.

We can say that we undermined the naïve acceptance of mathematics and put students in search of more knowledge.

GÖDEL'S INCOMPLETENESS THEOREM OPENING HORIZONS TO PHILOSOPHY OF MATHEMATICAL EDUCATION

The purpose of teaching GIT in teacher formation courses is to address the relationship between mathematics and logic, as well as to present and propose experiences that lead students to understand GIT relevance in the development of new methods and new ways of thinking in Modern Mathematics, the change this theorem brought about in the conception of the whole of mathematical knowledge and, consequently, to the conception of Mathematics itself. This transcends the knowledge of its statements and requires that the comprehensive work enables encounters with aspects of the result that may compose levels of knowledge of the ideas present in this theorem. As we understand, the presentation of intrinsic relations between mathematics and logic can occur by studying an illustrative version of Gödel's original proof. We aim to present the proposal that addresses the interrelationships between Mathematics and Logic through GIT study. The planned work of approaching Gödel's theorem, which demonstrates this result as a vehicle through we discuss metamathematics and the links between mathematics and logic, draws on teaching logic itself prevalent in mathematics teacher formation courses[6].

The most usual is the one that privileges the formalist conception and the structuralist systematization, aiming to show formally that there is mathematics knowledge as a social practice that cannot be part of a body structured in an axiomatized mathematical theory. We seek to elaborate an exercise in Philosophy of Mathematical Education that considers pertinent to explore and problematize, in formation courses to mathematics teachers, the historically produced concepts, ideas, processes and fundamental meanings of mathematics, rather than merely emphasizing its process of deductive axiomatic systematization without reflecting on it. Aiming at the implementation of this proposal - to work Gödel's incompleteness theorem in courses of formation of teachers of mathematics, presenting the links between Logic and Mathematics and also addressing the work with aspects of metamathematics - We claim that, aiming at its implementation in teaching, a "new" approach to research in Mathematical Education: (meta)mathematical education.

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[6] Our assertion concerns what happens in Brazil today, in general.

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